Past of a quantum particle: old problem with recent controversies

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Abstract

Time-symmetric formulation of quantum mechanics—the two-state vector formalism—is presented as a tool for studying past behaviour of quantum systems. A role of weak measurement and weak values in the Cheshire Cat effect and a nested (Vaidman) three-path interferometer are discussed. Interpretation of a particle's faint trace indicating possibility of discontinuous paths of particles passing the Vaidman interferometer is given. Consistent histories are presented as one of alternative approaches. Multitude of controversial issues is briefly reviewed and discussed.

Keywords

two-state vector formalism, weak values, weak measurement, Cheshire Cat effect.

1. Introduction

Copenhagen orthodoxy limits the predictive role of quantum mechanics to measurement results only and introduces unavoidable indeterminacy at a quantum level. One can prepare (preselect) a quantum system, evolve it (via e.g. unitary time evolution) and then one can measure it and postselect a particular, desired result. The mea-

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surement output is known to belong to a certain and well defined set of possible values which occur, as an output, with certain probability. Such an approach is already more than century old and we are used to an uncomfortable fact that at a quantum level not only measurement results but also quantum measurement *per se* require an interpretation.

In a standard von Neumman's scheme a measurement output of an observable is one of eigenvalues a_k of a hermitian operator A representing the observable i.e. $A|a_k\rangle = a_k|a_k\rangle$ with the corresponding eigenvectors $|a_k\rangle$ forming a basis spanning a state space of the system. Let us forewarn that for a simple convenience we use notation suitable for qudits (finite d–dimensional quantum systems) rather than general quantum systems. Such a convention allows one to bound an applied formalism almost solely to linear algebra at a low cost of a little mathematical 'flexibility' required for studying infinite systems. Having a system in a state $|\psi\rangle = \sum_k c_k|a_k\rangle$ a probability of measuring a_k is given by a scalar product of the state and the particular eigenstate $|a_k\rangle$ and reads $prob(k) = |\langle a_k|\psi\rangle|^2$. We humbly acknowledge that one cannot say *before* the measurement is done which among a_k's is going to be observed and we are left with an expectation only: $\langle A\rangle = \langle\psi|A|\psi\rangle = \sum_k p_k a_k$. Moreover, we know that such a measurement procedure causes a serious disturbance of a system as it results in a jump or collapse of the state $|\psi\rangle \to |a_k\rangle$ which can be formalised in terms of projection operators $|a_k\rangle\langle a_k|$.

In more formal terms, with an applied notation summarized in Tab.(1), one prepares a composite state of a system and a probe (a pointer) $|\Psi(t_i)\rangle = |\psi(t_i)\rangle|\phi(t_i)\rangle$. To measure a system, the probe must, at least for a certain duration τ, interact with the measured object. Measuring an observable A one assumes that the system–probe interaction is given by a Hamiltonian coupling A and a 'momentum' (an operator canonically conjugated to the probe position)

states of	applied notation
system+probe	$\|\Psi\rangle$ (capital)
system preselected	$\|\psi\rangle$, $\|a\rangle$
system postselected	$\|b\rangle$
probe only	$\|\phi\rangle$

Table 1: Notation of states applied in the paper with time parameter and subscripts skipped.

$H_{int} = g(t)AP$ with a coupling strength $g = \int_{t_i}^{t_i+\tau} g(t)dt$. Hence, the von Neumann ideal measurement is nothing but a dynamic process

$$\begin{aligned} |\Psi(t_i+\tau)\rangle =& e^{-igAP}|\psi(t_i)\rangle|\phi_{x_0}\rangle \\ =& \sum_k |a_k\rangle\langle a_k|\psi(t_i)\rangle e^{-iga_kP}|\phi_{x_0}\rangle \\ =& \sum_k |a_k\rangle\langle a_k|\psi(t_i)\rangle|\phi_{x_0-ga_k}\rangle \end{aligned} \tag{1}$$

causing shift of a pointer position from x_0 (in the state $|\psi_{x_0}\rangle$ to $x_0 - ga_k$ for $|\phi_{x_0-ga_k}\rangle$. The last step, i.e. a projective measurement of the pointer, allows one to relate a particular read-out value $x_0 - ga_k$ to a particular value a_k of the observable A.

It is not surprising that there are circumstances where limitations of Copenhagen interpretation (and the celebrated eigenvalue–eigenstate link) become quite bothersome. We present here only one of many problems: a measurement result allows one to gain knowledge about the present of quantum systems (a particle). It is, however, hardly possible to infer upon what was the particle's *past*, what 'historic' properties one can attribute to the just measured systems. Sometimes it is even harder to say anything about quantum systems which are essentially closed (as, for example, the whole universe is) and do not allow to include an external observer keeping a pointer or these which are simply resistant to measuring attempts. Neutrino, hardly interacting with anything, can serve as a natural example of such a 'measurement–resistive' system. Counterfactual reasoning (what would happen if a particular measurement were performed or, more generally, if there were that a, then it would be that b) is related to an another branch of problems where the eigenvalue–eigenstate link induces unwanted constraints.

Our aim is to present a seemingly simple quantum system—a particle in a nested three-path interferometer invited by Lev Vaidman to a modern scientific debate. A particle enters the interferometer, it has three paths at its disposal and then it is measured (postselected) leaving the interferometer by one of three exits. The question which is asked looks innocent and apparently simple: what path took the particle inside the interferometer? What was its trajectory? The questions are natural (particles entering and leaving the interferometer must somehow pass it) and interesting (cf. reference list below) although from the Copenhagen perspective slightly out of place. In this paper we focus on the two-state vector formalism recently applied to answer the above question and leading to hardly acceptable and very disputable possibility: the particle in the Vaidman interferometer follows a *discontinuous* path. Surprisingly, such a weird answer, seemingly contradicting not only our intuition but also common sense, is supported by very recent experiments and as such requires careful analysis not only from a physical but also a more fundamental, philosophical perspective. Although the two-state vector formalism is nothing but a time-symmetric version of a very standard quantum mechanics, it exhibits its true predictive power combined with a notion of *weak measurement* and *weak value* of an observable. Both concepts, exemplified with help of the Cheshire Cat effect, are lacking from the eigenvalue-eigenstate link and require an interpretation. The paper is organized as follows: in Sec. 2 the two-state vector formalism is presented, a notion of weak measurement and weak values are applied to present the Cheshire Cat effect of a spatial separation of a photon position and polarization. Sec 3. is devoted to the Vaidman's construction of a nested interferometer and a faint trace as a tool verifying the presence of a weakly measured particle inside the interferometer. As the two-state formalism is not the only formalism applied to study particle's past in the Vaidman interferometer, in Sec 4. we very briefly present one of alternatives: an approach based on consistent histories and we indicate probably the most obvious and easiest to formulate objection against this approach. Concluding in Sec 5. we list selected problems which—at least in our opinion—urgently require a professional philosophical analysis.

2. Two–state vector formalism and the Cheshire Cat

Classical dynamics of a particle is a solution of the Newton equation which, to uniquely predict a final state, needs initial conditions (an initial state) to be specified. For quantum system to predict a measurement output at a given time it is not enough to preselect an initial state and then to solve the Schrödinger equation as a resulting final state may be a superposition. To get unique prediction of a measurement output one needs an additional information what state was postselected at the end of evolution. In other words, in classical physics all the results of future measurements are constrained by the results of the past measurements leading to an initial preparation. For quantum systems, however, a measurement output is only partially determined by past measurements and we recognize that quantum mechanics, contrary to the classical one, is essentially time asymmetric. This asymmetry limits an ability of reproducing the past solely upon the present measurement results. The two-state vector formalism (Aharonov and Vaidman, 2008) (TSVF) is an attempt to make quantum mechanics time-symmetric by adding an information concerning a postselected state. According to the TSVF, to describe a quantum system at a time t one needs a two-state vector

$$\langle\omega(t)|\ |\alpha(t)\rangle \tag{2}$$

where $|\alpha(t)\rangle$ is a 'usual' state of a time-evolving quantum system which is a fundamental object of a standard quantum mechanics and $\langle\omega(t)|$ is a state evolving *backward in time* and determined by the results of measurements performed on the system *after* the time t. In other words, the two-state vector is pair of states consisting of a forward evolving preselected state and a backward evolving postselected state. Let us imagine that at $t = t_i$ our system is in one of eigenstates of an observable A i.e. $|\alpha(t_i)\rangle = |A = a\rangle$ and finally, at $t = t_f$, $\omega(t_f)\rangle = |B = b\rangle$ for a different observable B. According to the TSVF a complete information about the system at $t \in (t_i, t_f)$ is encoded in $\langle\omega(t)|\ |\alpha(t)\rangle$ where

$$\begin{aligned} |\alpha(t)\rangle &= U(t, t_i)|A = a\rangle \\ |\omega(t)\rangle &= U(t, t_f)|B = b\rangle \end{aligned} \tag{3}$$

and $U(\cdot,\cdot)$ is a unitary time evolution operator. Let us strongly emphasise that essentially the TSVF is a standard quantum mechanics just equipped with an additional information carried by a backward evolving state. As such, the TSVF is not expected to contradict any 'standard' predictions of the quantum mechanics. Minimalists may consider TSVF just as a useful tool to analyze otherwise strange and 'paradoxical' phenomena.

Let us describe the following scenario to present a rationale behind the TSVF construction and to show a particular usefulness of this formalism in a weak measurement context. The Aharonov-Bergmann-Lebowitz rule (Aharonov, Bergmann and Lebowitz, 1964) is a working horse of many of subsequent techniques. If there is a quantum system preselected in a state $|\psi(t_i)\rangle$ and finally found in a postselected state $|b_f\rangle$ conditioned probability that an observable $A = \sum_n a_n |a_n\rangle\langle a_n|$ measured at some intermediate stage gives a_n reads as follows:

$$prob(a_n) = \frac{|\langle b_f|a_n\rangle\langle a_n|\psi(t_i)\rangle|}{\sum_s |\langle b_f|a_s\rangle\langle a_s|\psi(t_i)\rangle|^2}. \tag{4}$$

We consider a narrow time window $t_w \pm \tau/2$ when the pointer–system interaction takes place. Out of this window a time evolution of a system+probe composite is unitary and governed by U. Then, a time evolved system–pointer state reads as follows:

$$\begin{aligned} |\Psi(t)\rangle =& U(t, t_w)e^{-gAP}U(t_w, t_i)|\psi(t_i)\rangle|\phi(t_1)\rangle \\ =& U(t, t_w)e^{-igAP}|\psi(t_w)\rangle|\phi(t_1)\rangle \\ =& U(t, t_w)\sum_k e^{-iga_kP}\langle a_k|\psi(t_w)\rangle|a_k\rangle|\phi(t_i)\rangle \end{aligned} \tag{5}$$

Finally, at $t = t_f$, the system becomes postselected i.e. it is subjected to an (ideal) projective measurement of an observable B reducing its state to of eigenstates $|b_f(t_f)\rangle$ of B. Upon the Aharonov-Bergmann-Lebowitz rule the pointer after the system postselection is left in a state

$$|\phi(t_f) = \sum_k [\langle b_f(t_f)|U(t_f, t_w)|a_k\rangle\langle a_k|\psi(t_w)\rangle]e^{-iga_kP}|\phi(t_i)\rangle \tag{6}$$

Let us note that, at least formally, $\langle b_f(t_w)| = \langle b_f(t_f)|U(t_f, t_w)$ is a bra-vector which is dual to a state $|b_f(t_w)\rangle = [\langle b_f(t_w)|]^\dagger =$

$[\langle b_f(t_f)|U(t_f,t_w)]^\dagger = U^\dagger(t_f,t_w)|b_f(t_f)\rangle = U(t_w,t_f)|b_f(t_f)\rangle$ which, in turn, is a postselected system state evolving *backward* in time. If in addition the system-probe interaction is weak—i.e. $g \ll 1$— one can expand $e^{-iga_kP} \approx 1 - iga_kP$ and obtain

$$
\begin{aligned}
|\phi(t_f)\rangle =& \langle b_f(t_w)|\psi(t_w)\rangle \left(1 - ig\langle A\rangle^w P\right)|\phi(t_i)\rangle \\
=& \langle b_f(t_w)|\psi(t_w)\rangle \exp(-ig\langle A\rangle^w P)|\phi(t_i)\rangle
\end{aligned}
\tag{7}
$$

where the quantity

$$
\langle A\rangle^w = \frac{\langle b_f(t_w)|A|\psi(t_w)\rangle}{\langle b_f(t_w)|\psi(t_w)\rangle}
\tag{8}
$$

defines a *weak value* (Aharonov, Albert and Vaidman, 1988) of A. The weak value is thus a number characterizing a shift (or a motion) of a quantum pointer due to a weak interaction with a measured system. Although the weak value is generically complex, it is a measurable quantity (Dressel et al., 2014) and its real part can be interpreted as an amplitude of a measurement induced transition from a preselected into postselected state. Its squared modulus gives the corresponding probability of the process. Such an interpretation settles weak values in a natural context of a measurement driven dynamic process.

Quantum weak values (Aharonov, Albert and Vaidman, 1988) attract non–decreasing attention of physicists who apply them to analyze otherwise difficult problems. Let us mention only two examples: continuous and sometimes controversial discussion what the history of quantum particle really is (Englert et al., 2017; Vaidman, 2013a; 2014) or one of the spectacular counter-intuitive effects—the quantum Cheshire Cat (Aharonov, Popescu et al., 2013). Weak values seem to be of a particular usefulness in all the circumstances which require simultaneous measurement of otherwise non co–measurable observables (Aharonov, Cohen, Waegell et al., 2018; Aharonov, Popescu et al., 2013; Aharonov and Vaidman, 2008; Dressel et al., 2014; Vaidman, Ben-Israel et al., 2017). Let us emphasise one but crucial feature concerning *null* weak values. In a well established scheme of an ideal measurement Eq.(1) the pointer state couples to a particular eigenvector with a particular eigenvalue a_k. If $a_k = 0$ there is no effect on a pointer state and the probe remains not 'shifted'. There is however an obvious modification of the system state which is projected on a subspace corresponding to the null eigenvalue. In the weak measurement scheme the situation essentially differs. If a particular *weak value* vanishes $\langle A\rangle^w = 0$ and a postselected state of a system is obtained the property corresponding to A cannot be detected. The weak value indicates a coupling-induced imprint which is left on the pointer, conditioned on the postselection. As a consequence, the vanishing weak value correlates successful postselection with the quantum pointer having been left unchanged despite the interaction with the system. Let us emphasise that for weak values Eq.(8) there is no clear analogy of the eigenvalue-eigenstate link and their relation to quantum properties remain a subject of an intensive investigation (Aharonov, Cohen, Waegell et al., 2018; Matzkin, 2019; Vaidman, Ben-Israel et al., 2017).

To explore further an interpretation of null weak value (further used in a general context of a past of quantum particles (Duprey and Matzkin, 2017; 2018; Sokolovski, 2018)) we discuss one of the first quantum phenomena exhibiting non–intuitive predictions based upon weak values and weak measurement scheme: the Cheshire Cat effect (Aharonov, Popescu et al., 2013; Ashby, Schwarz and Schlosshauer, 2016). For the sake of making our discussion as self–contained as possible we recall a very basic idea of the Cheshire Cat as primarily introduced in (Aharonov, Popescu et al., 2013). The quantum Cheshire Cat effect indicates separation of internal and exter-

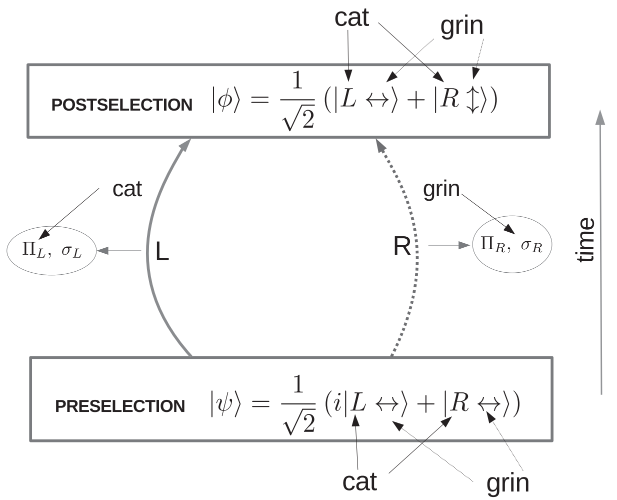

Figure 1: The Cheshire Cat effect. Photon in an interferometer is preselected in a state $|\psi\rangle$. It is then *weakly* measured along left and right arms (possible L, R paths) and then postselected in a state $\phi\rangle$. The weak values $\langle\Pi_L\rangle^w = 1$, $\langle\Pi_R\rangle^w = 0$, $\langle\sigma_L\rangle^w = 0$ and $\langle\sigma_R\rangle^w = 1$ interpreted as a presence of the Cat in the left arm and its grin in the right one indicate separation of photon polarization ('grin') and photon position.

nal degrees freedom of a quantum system corresponding to the Cat's position and the grin respectively and justifying a direct implementation of terminology adopted from Carroll's novel *Alice's Adventures in Wonderland*. The archetype of the Cheshire Cat (Aharonov, Popescu et al., 2013) is a photonic system and an effect takes place in a *two–path* Mach-Zehnder setting with a path 'chosen' by the photon—either left L or right R—describing the 'external' degree of freedom and photonic polarization—horizontal $\leftrightarrow$ or vertical $\updownarrow$ representing internal degree of freedom as it is presented in Figure 1 above or on Figure 1 in (Aharonov, Popescu et al., 2013). Effectively, both types of degree of freedom are qubits i.e. a state space of the system is $\mathcal{H} = \mathbf{C}^2 \otimes \mathbf{C}^2 = \text{span}\{|L\rangle, |R\rangle\} \otimes \text{span}\{|\leftrightarrow\rangle, |\updownarrow\rangle\}$ where L, R and $\leftrightarrow, \updownarrow$ label the 'external' and internal degrees of freedom respectively. Detection of the Cat's position corresponds to a measurement related to the projectors: $\Pi_L = |L\rangle\langle L|$ and $\Pi_R = |R\rangle\langle R|$ whereas a measurement of Cat's grin (an internal degree of freedom) in a given (either left or right) position requires the projectors $\sigma_L = \Pi_L \sigma_z$ and $\sigma_R = \Pi_R \sigma_z$ where $\sigma_z = |+\rangle\langle+| - |-\rangle\langle-|$ for $|\pm\rangle = [|\leftrightarrow\rangle \pm i|\updownarrow\rangle]/\sqrt{2}$. According to the proposal given in (Aharonov, Popescu et al., 2013), the system is preselected (prepared) in a specific but not entangled state $|\psi\rangle = \frac{1}{\sqrt{2}}\left(i|L\leftrightarrow\rangle + |R\leftrightarrow\rangle\right)$ and, after passing the interferometer, postselected in a state $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|L\leftrightarrow\rangle + |R\updownarrow\rangle\right)$. In the meantime, one measures *weak values* Eq.(8) of the quantities $\Pi_{L,R}$ and $\sigma_{L,R}$ i.e. weak values of the Cat's position and grin respectively. One obtains (Aharonov, Popescu et al., 2013) $\langle \Pi_L \rangle^w = 1$, $\langle \Pi_R \rangle^w = 0$, $\langle \sigma_L \rangle^w = 0$ and $\langle \sigma_R \rangle^w = 1$. Whenever a weak value is null the corresponding quantum property (either the Cat's presence or its grin in one of two arm of the interferometer) is absent in a particular arm where the weak measurement was performed. With such an interpretation we infer that a presence of the Cat (indicated by $\langle \Pi_L \rangle^w \neq 0$) in the left L arm of the interferometer is accompanied by a presence of Cat's grin in the right R interferometer's arm as indicates $\langle \sigma_R \rangle^w \neq 0$. Clearly, the description provided here is far from being complete. In particular it does not take into account specific experimental circumstances typical for realistic Cheshire Cat measurements (Duprey, Kanjilal et al., 2018), recent experiments (Ashby, Schwarz and Schlosshauer, 2016; Denkmayr et al., 2014) or an effect of decoherence (Schlosshauer, 2007) modyfying weak values (Shikano and Hosoya, 2009) and the Cheshire Cat predictions (Dajka, 2020; Richter, Dziewit and Dajka, 2018). Moreover, interpreting null weak value as a hallmark of a '*non-*

presence of something' is controversial and far from being commonly accepted (Duprey and Matzkin, 2017; 2018; Hance, Rarity and Ladyman, 2021; Sokolovski, 2018; Vaidman, 2017b).

3. Faint trace of a particle in a Vaidman interferometer

Quantum particle can be prepared (preselected) in a given and desired state and may also be postselected in another state with a known at least in principle probability. That what occurs in between, what is the particle's past, remains problematic due to specific features of quantum measurements, cf. Eq.(1), unavoidably modifying quantum states of measured objects. It is particularly important if one asks about a history of a quantum particle passing through interferometer: the particle enters the device and leaves it (if its outcome is measured), however, inside the interferometer, unless it's coherence is lost, the particle leaves nothing but a faint trace *defining* its presence, a faint trace which is a result of a *small* change of an amplitude of a component orthogonal to an undisturbed particle's state (weakly). Such a faint trace is measurable only in experiments operating on an ensemble of particles having the same pre– and postselection. Past of a quantum particle in a nested Mach–Zehnder interferometer (*Vaidman interferometer*)—proposed (Vaidman, 2013a) and presented in Figure (2)—was recently studied in (Vaidman, 2013a) using quantum weak values (Aharonov, Albert and Vaidman, 1988; Aharonov and Vaidman, 2008; Vaidman, Ben-Israel et al., 2017) and the two state vector formalism (TSVF) (Aharonov and Vaidman, 2008). In this approach the faint trace is left by a particle unless a *weak value* of an appropriate projecting operator vanishes. A weak measurement

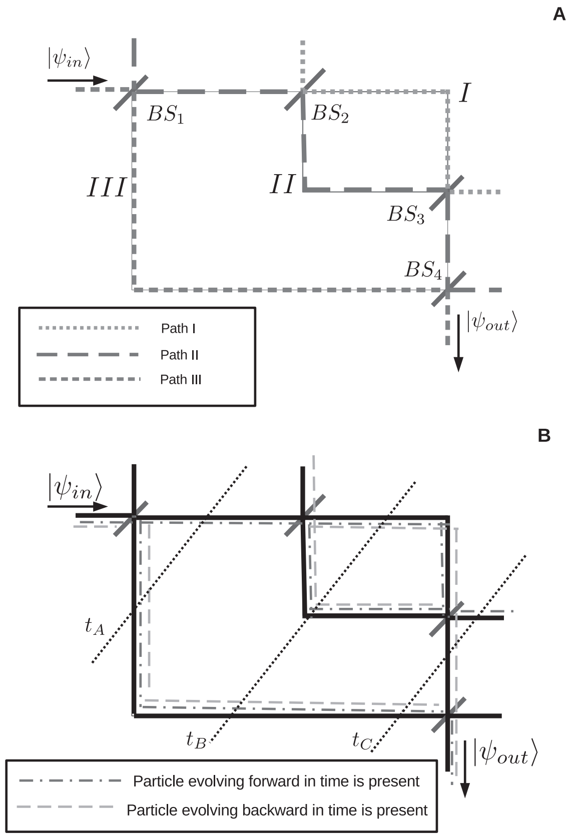

Figure 2: Vaidman interferometer consisting of four beam splitters $BS_{1,2,3,4}$ acting as unitary transformation in Eq.(10). In *panel A* there are three 'paths' Eq.(9) indicted I, II, III and labelled by different colours. An entrance of an input a particle in a state $|\psi_{in}\rangle$ and an exit of a particle in a state $|\psi_{out}\rangle$ are indicated by arrows. Time instants $t_{A,B,C}$ when weak measurements take place are indicated by dotted lines in *panel B*. The dash-dot and dash lines shown in a legend box of *panel B* indicate a *non-zero* amplitude for meeting a particle evolving forward and (respectively) backward in time. According to the TSVF the particle leaves a faint trace only if *both lines* coincide i.e. on a green path (III) and inside the internal interferometer as it is indicated in *panel B*.

of particle traces, contrary to a traditional and collapse assisted one, hardly perturbs the system maintaining its coherence sufficiently for interference effects to occur. The results, however, are highly confounding: particles seem to follow anomalous *discontinuous* path. Such a seemingly weird conclusion results in plethora of controversies (Li, Al-Amri and Zubairy, 2013; Vaidman, 2013b), some of them are quite recent (cf. Hance, Rarity and Ladyman, 2021), and since that time (almost) all works on that problem have came in triads: a paper, commentary inspired by the paper and a reply to the comment (Li, Al-Amri and Zubairy, 2013; Vaidman, 2013b). The main reason is that the TSVF (Aharonov and Vaidman, 2008) applied in (Vaidman, 2013a) is one of few possible approaches to investigate quantum past. The other non–equivalent alternatives are consistent (decoherent) histories (Griffiths, 2016; Vaidman, 2017a) (briefly presented below), standard quantum mechanics (Englert et al., 2017; 2019; Peleg and Vaidman, 2019) and many other other alternative studies (Aharonov, Cohen, Landau et al., 2017; Bartkiewicz et al., 2015; Hashmi et al., 2016; 2018; Potoček and Ferenczi, 2015; Vaidman, 2016a,b; 2018). Moreover, even recent experiments and their detailed analyses fail to resolve all the controversial issues (Aharonov, Cohen, Landau et al., 2017; Danan et al., 2013; Geppert-Kleinrath et al., 2018; Saldanha, 2014; Salih, 2015; Sponar et al., 2019; Vaidman, Danan et al., 2015; Wieśniak, 2018; Zhou et al., 2017). A conception of discontinuous path is clearly counter–intuitive but there are analyses (Aharonov, Cohen, Waegell et al., 2018; Vaidman, 2020) and claims which support the faint–trace anomalous picture as experimentally confirmed.

For a sake of completeness we formalize the Vaidman interferometer and review the controversial features of the faint traces of particles passing it. The original Vaidman interferometer is presented in Figure (2). It consists of spatial degree of freedom given by three paths denoted by I, II, III respectively and four beam splitters. The Vaidman interferometer can effectively be described (Dajka, 2021; Englert et al., 2017; 2019; Peleg and Vaidman, 2019) as a three level quantum system (a qutrit) with a state space spanned by an orthonormal basis

$$|I\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \ |II\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \text{ and } \ |III\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{9}$$

Let us note an intuitively acceptable rationale behind imposing orthogonality on the set Eq.(9): if a particle collapses to a particular state in Eq.(9), at the same time there is no amplitude to be in an another one. In an ideal setting of a noise–less system (for a noisy dephasing model (cf. Dajka, 2021)), a passage of a particle is described by a unitary transformation composed of four unitaries $U_4U_3U_2U_1$ corresponding to subsequent beam splitters termed as $BS_i, i = 1, \ldots, 4$ in Figure 2:

$$\begin{aligned} U_1 = U_4 &= \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{3} & 0 & 0 \\ 0 & -1 & \sqrt{2} \\ 0 & \sqrt{2} & 1 \end{pmatrix} \text{ and} \\ U_2 = U_3 &= \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}. \end{aligned} \tag{10}$$

The strategy applied in (Vaidman, 2013a) to infer the path of a particle entering and leaving Vaidman interferometer in a state $|III\rangle$ was to investigate its *weak trace* at three instants t_A, t_B, t_C indicated in Figure (2). In a framework of the TSVF (Aharonov and Vaidman, 2008) and according to (Vaidman, 2013a) the weak trace is indicated by a non–vanishing weak value (Aharonov, Albert and Vaidman, 1988; Duprey and Matzkin, 2017) of one of the projectors

$$\langle \Pi_i \rangle_w^q = \frac{\langle \psi^q_{post} | \Pi_i | \psi^q_{pre} \rangle}{\langle \psi^q_{post} | \psi^q_{pre} \rangle}, \text{where} \;\; \Pi_i = |i\rangle\langle i|, \;\; i = I, II, III \quad \text{and} \;\; q = A, B, C \tag{11}$$

where forward in time evolving preselected state (directly prior to the measurement of Π_i) and backward in time postselected state (immediately after the measurement) states compose a *two–state vector* $\langle \psi_{post} || \psi_{pre} \rangle$.

Most of controversies originate from highly counter–intuitive conclusions provided in (Vaidman, 2013a) indicating possibility of discontinuous trajectories followed by a particle passing trough Vaidman interferometer. There are three instants t_A, t_B, t_C where the weak trace is measured: A: just after it passes the interferometer BS_1 but before it arrives to BS_2, B: where the weak measurement becomes conducted for all three potential paths and C: after the BS_3 beam splitter as presented in Figure (2). The corresponding forward-in-time evolving preselected states read as: $|\psi^A_{pre}\rangle = U_1|III\rangle$, $|\psi^B_{pre}\rangle = U_2U_1|III\rangle$ and $|\psi^C_{pre}\rangle = U_3U_2U_1|III\rangle$. In the time–symmetric TSVF setting the postselected states $|\psi^A_{post}\rangle = U_2^\dagger U_3^\dagger U_4^\dagger |III\rangle$, $|\psi^B_{post}\rangle = U_3^\dagger U_4^\dagger |III\rangle$ and $|\psi^C_{post}\rangle = U_4^\dagger |III\rangle$ describe a hypothetical particle detected at III evolving backward in time. Results of the weak measurement are summarized in Table (2). According to (Vaidman, 2013a) a presence of a particle is defined by its non–vanishing weak trace. The counter–intuitive conclusion which can be inferred upon the results summarized in Table (2) is the following: at A and C the particle is present in III, what upon Figure (2) is intuitively acceptable, but at B it is also present in an internal loop (I, II) of the Vaidman interferometer. The above formal analysis one can support utilizing the

$\langle\Pi_{I,II,III}\rangle_w^{A,B,C}$	I	II	III	$U_{pre}^{A,B,C}$	$U_{post}^{A,B,C}$
A	0	0	1	U_1	$U_2^\dagger U_3^\dagger U_4^\dagger$
B	-1	1	1	U_2U_1	$U_3^\dagger U_4^\dagger$
C	0	0	1	$U_3U_2U_1$	$U_4^\dagger$

Table 2: Weak traces $\langle\Pi_{I,II,III}\rangle_w^{A,B,C}$ Eq.(11) of a particle in Vaidman interferometer in Figure (2) at different instants A, B, C indicated in Figure (2) and corresponding pre– and postselections given by $|\psi_{pre}^{A,B,C}\rangle = U_{pre}^{A,B,C}|III\rangle$ and $|\psi_{post}^{A,B,C}\rangle = U_{post}^{A,B,C}|III\rangle$ respectively. It follows that at t_A and t_C a particle leaves its weak trace on paths I and III respectively whereas at t_B a faint trace is present on each of three paths.

TSVF and a *sine qua non* condition for a presence of particle: to get a non-vanishing *weak value* of a particular projector in a weak measurement scheme both the amplitudes of forward and backward evolving component of the two–vector necessarily must not vanish. The regions where the amplitudes of forward and backward evolving states do not vanish are depicted in Figure (2) with dash-dot and dash lines respectively. The only regions where the lines coincide are the path labeled by III and the internal loop of the Vaidman interferometer. There is a faint trace left by particles on the path II neither between beam splitters BS_1 and BS_2 (potentially used photons entering the internal loop) nor between BS_3 and BS_4 where the photons could exit the internal loop. In simple words, upon the TSVF we conclude that a presence of a particle indicated by a faint trace in an internal loop is not accompanied by any trace of a particle entering or leaving the internal loop and the particle path is *discontinuous*. It is obvious that such a confounding result needs further experimental verification. One can safely assume that any potential experiment, as it was so far, will be highly subtle and sophisticated (Aharonov, Cohen, Waegell et al., 2018; Dziewior et al., 2019; Rebufello et al., 2021; Vaidman, 2020).

4. Consistent histories as an alternative

Orthodox (Copenhagen orthodox) researchers claim that one cannot talk about a quantum particle between measurements at all. This is an obvious limitation of standard quantum theory radically excluding important questions related to a past behaviour of quantum systems. Previously discussed two–state vector formalism (Aharonov and Vaidman, 2008) and consistent (or decoherent) histories approach (Gell-Mann and Hartle, 1993; 1999; Griffiths, 1984; 2003; Omnès, 1988; 1994) serve as fruitful examples of theoretical extensions going beyond (and sometimes across) the Copenhagen interpretation. In particular, we have recently been witnessing how the two above-mentioned approaches are competitively applied to a problem of a past behaviour of a quantum particle in a Vaidman interferometer (Dajka, 2021; Vaidman, 2013a; 2017a). Using different methods, consistent histories allow one to gain an additional insight if they are applied to problems ranging from a small but fundamental (Griffiths, 2013; 2014; 2015; 2017) to the largest scale (Craig, 2016; Riedel, Zurek and Zwolak, 2016). The consistency of histories allows one to assign probabilities to sequences of suitably defined events for a quantum system. As the quantum events in this perspective do not rely on the notion of measurement *per se*, the consistent histories approach enables one to design a new type of logical approach (Griffiths, 1984) which is essentially different to the Birkhoff and von Neumann quantum logic (Birkhoff and Von Neumann, 1936). There is a particular practical advantage of using consistent histories formalism to gain information about a system if an external measurement is not available either because of fundamental or simply technical reasons as it is in the case of Vaidman interferometer where an experiment output is highly sensitive to a measurement-induced coherence deficiency. Quantum reasoning based on consistent histories (Griffiths, 1996) uses projective decomposition of identity $PDI = \{P^k\}$ where $P^j P^k = \delta_{jk} P^k$ and $\sum_j P^j = I$ as its cornerstone (Griffiths, 2003). It serves as a quantum-mechanical counterpart of an event algebra used in standard stochastics. There is, however, one crucial yet fundamental requirement which is additionally imposed: *the single framework rule* (Griffiths, 1996; 2003; 2015). According to that rule simultaneous reasoning to physical properties is meaningful if it is limited to 'events' which are compatible (Griffiths, 1996; 2003; 2013; 2015). At each time instant all quantum properties of a system correspond to elements of an instantaneous PDI having assigned probabilities and a time evolution of the quantum systems studied with consistent histories model can be considered as a stochastic process. With such an approach neither the future nor the past states of the system must be determined by the present state since, instead, they are (only) related by their probabilities. If the probabilities are 0 or 1 one arrives at a deterministic time evolution. A mathematical stage accommodating sequences of events is a tensor product of 'instantaneous' Hilbert spaces. Quantum properties (events) related to time evolving systems are its *histories* forming time–dependent PDIs where (generically) $PDI(t_i) \neq PDI(t_j)$ for different time instants giving one a chance to pose different questions concerning different quantum properties of the systems at different time instants. Assigning probabilities to non-commuting quantum properties is only meaningful if there is no interference between pairs of histories which suppose to be *decoher-*

ent. After extending the celebrated Born rule to multi–time history one can assign a weight to a sequence of events (Griffiths, 1984; 1996; 2003; 2013; 2015) and impose *consistency condition* (Griffiths, 1984; 1996; 2003; 2013; 2015) satisfied by histories which are meaningful. Let us note that a quantum history of a physical system is a sequence of quantum events at successive times, where a quantum event at a particular time can be any quantum property of the system in question (Griffiths, 2003). Such a convenient tool can serve to analyse properties of systems that are very difficult to measure and, in particular, consistent reasoning has already been applied to study history of a particle in a Vaidman interferometer. The result was that the TSVF predictions described above are based upon inconsistent histories and hence are meaningless. Clearly, an associate debate (Griffiths, 2016; Vaidman, 2017a) was surprising to nobody. Consistent histories seem to be an attractive and mathematically sound extension of quantum mechanics (or quantum stochastics) also suitable to study the past of quantum systems or even to dissolve the (in)famous measurement problem (Griffiths, 1996; 2015). The single framework rule, however, crucial for consistent reasoning, is highly more 'invasive' for quantum theory in comparison to a simple inclusion of a backward in time evolving postselected state as it is done in the TSVF. To avoid long and technical argumentation to support this statement and to indicate both existing and potential problems with the consistent histories let us invoke Mermin's opinion from (2013):

> [But] I am disconcerted by the reluctance of some consistent historians to acknowledge the utterly radical nature of what they are proposing. The relativity of time was a pretty big pill to swallow, but the relativity of reality itself is to the relativity of time as an elephant is to a gnat.

5. Concluding (yet not conclusive) remarks

Weak values, contrary to standard eigenvalues, are still waiting for a commonly accepted interpretation. In an absence of the eigenstate-eigenvalue link it is not obvious if weak measurements and their outputs can credibly describe ‘elements of reality’ and properties of quantum systems (Matzkin, 2019; Vaidman, 2017b). It is agreed that weak values possess certain non–trivial predictive value and they are related, at least to some extent, to real properties of physical systems. Unfortunately, it is difficult to assign any ‘hard’ limits of their applicability. Life would be definitely simpler if weak values were strongly measurable. In particular, physical meaning of vanishing weak values applied in a current context of a past of quantum systems remains disputable (Duprey and Matzkin, 2017; 2018; Hance, Rarity and Ladyman, 2021; Sokolovski, 2018; Vaidman, Ben-Israel et al., 2017) although there are analyses and experiments (Aharonov, Cohen, Waegell et al., 2018; Dziewior et al., 2019; Rebufello et al., 2021; Vaidman, 2020) which support the faint–trace anomalous picture as experimentally confirmed.

Time-symmetric description of quantum systems utilizing the TSVF (Aharonov, Cohen and Landsberger, 2017; Aharonov and Vaidman, 2008), despite certain problematic issues concerning its adaptation to open quantum systems requiring mixed states for their description (Dajka, 2021; Vaidman, Ben-Israel et al., 2017), seems to be less controversial. At the same time, however, it affects new research areas such as counterfactual reasoning (Vaidman, 1999) or even the hot-forever *free will problem* (Aharonov, Cohen and Shushi, 2016).

Counterfactuality and counterfactual reasoning (‘if it were a, then it would be b’) (Lewis, 2001; Vaidman, 1999) is a natural extension of an interaction-free measurement, cf. Elitzur and Vaidman’s interaction-free bomb detector (Elitzur and Vaidman, 1993), where a bomb (a detector, using more pacifistic terminology) indicating a presence of a single photon is put in one of the arms of a Mach–Zehnder interferometer. Even if the bomb does not blow up, its presence affects an interference pattern at the output of the interferometer. Direct application of the Aharonov-Bergman-Lebowitz rule Eq.(4) for counterfactual reasoning is not always sufficient and acceptable (Vaidman, 1999). The idea of using a time-symmetric approach to quantum counterfactuals has recently found a promising application in quantum-based counterfactual communication. Such communication protocols are counterfactual by using quantum effects to send messages without any matter or energy transfer between communicating parties (Vaidman, 2019; Wander, Cohen and Vaidman, 2021)). There are already known crypto protocols (Hance, Ladyman and Rarity, 2021; Kamaruddin, Shaari and Kolenderski, 2020; Noh, 2009; Rao and Srikanth, 2021) based upon that idea. Obviously, all the controversies concerning weak values, past of quantum systems and counterfactual reasoning both mentioned and not mentioned in this work to be resolved need to be supported by further experimental investigations.

One can conclude that with a tremendous development of highly sophisticated experimental techniques we are faced (maybe for the first time) with problems which *simultaneously* require advanced technology, fresh and flexible theory and, last but not least, sound interpretation. Let us take up this challenge.

Bibliography

Aharonov, Y., Albert, D.Z. and Vaidman, L., 1988. How the Result of a Measurement of a Component of the Spin of a Spin-1/2 Particle Can Turn Out to Be 100. *Physical Review Letters* [Online], 60(14), pp.1351–1354. https://doi.org/10.1103/PhysRevLett.60.1351.

Aharonov, Y., Bergmann, P.G. and Lebowitz, J.L., 1964. Time Symmetry in the Quantum Process of Measurement. *Phyical Review* [Online], 134(6B), B1410–B1416. https://doi.org/10.1103/PhysRev.134.B1410.

Aharonov, Y., Cohen, E., Landau, A. and Elitzur, A.C., 2017. The Case of the Disappearing (and Re-Appearing) Particle. *Scientific Reports* [Online], 7, p.531. https://doi.org/10.1038/s41598-017-00274-w.

Aharonov, Y., Cohen, E. and Landsberger, T., 2017. The Two-Time Interpretation and Macroscopic Time-Reversibility. *Entropy* [Online], 19(3). https://doi.org/10.3390/e19030111.

Aharonov, Y., Cohen, E. and Shushi, T., 2016. Accommodating Retrocausality with Free Will. *Quanta* [Online], 5(1), pp.53–60. https://doi.org/10.12743/quanta.v5i1.44.

Aharonov, Y., Cohen, E., Waegell, M. and Elitzur, A.C., 2018. The Weak Reality That Makes Quantum Phenomena More Natural: Novel Insights and Experiments. *Entropy* [Online], 20(11). https://doi.org/10.3390/e20110854.

Aharonov, Y., Popescu, S., Rohrlich, D. and Skrzypczyk, P., 2013. Quantum Cheshire Cats. *New Journal of Physics* [Online], 15(11), p.113015. Available at: <http://stacks.iop.org/1367-2630/15/i=11/a=113015>.

Aharonov, Y. and Vaidman, L., 2008. The Two-State Vector Formalism: An Updated Review. In: J. Muga, R.S. Mayato and Í. Egusquiza, eds. *Time in Quantum Mechanics* [Online]. Berlin, Heidelberg: Springer, pp.399–447. https://doi.org/10.1007/978-3-540-73473-4_13.

Ashby, J.M., Schwarz, P.D. and Schlosshauer, M., 2016. Observation of the Quantum Paradox of Separation of a Single Photon from One of Its Properties. *Physical Review A* [Online], 94(1), p.012102. https://doi.org/10.1103/PhysRevA.94.012102.

Bartkiewicz, K. et al., 2015. One-State Vector Formalism for the Evolution of a Quantum State Through Nested Mach-Zehnder Interferometers. *Physical Review A* [Online], 91(1), p.012103. https://doi.org/10.1103/PhysRevA.91.012103.

Birkhoff, G. and Von Neumann, J., 1936. The Logic of Quantum Mechanics. *Annals of Mathematics* [Online], 37(4), pp.823–843. https://doi.org/10.2307/1968621.

Craig, D.A., 2016. The Consistent Histories Approach to Loop Quantum Cosmology. *International Journal of Modern Physics D* [Online], 25(08), p.1642009. https://doi.org/10.1142/S0218271816420098.

Dajka, J., 2020. Scattering–Like Control of the Cheshire Cat Effect in Open Quantum Systems. *Quantum Reports* [Online], 2(1), pp.1–11. https://doi.org/10.3390/quantum2010001.

Dajka, J., 2021. Faint Trace of a Particle in a Noisy Vaidman Three-Path Interferometer. *Scientific Reports* [Online], 11(1), p.1123. https://doi.org/10.1038/s41598-020-80806-z.

Danan, A., Farfurnik, D., Bar-Ad, S. and Vaidman, L., 2013. Asking Photons Where They Have Been. *Physical Review Letters* [Online], 111(24), p.240402. https://doi.org/10.1103/PhysRevLett.111.240402.

Denkmayr, T. et al., 2014. Observation of a Quantum Cheshire Cat in a Matter-Wave Interferometer Experiment. *Nature Communications* [Online], 5, p.4492. Available at: <http://dx.doi.org/10.1038/ncomms5492>.

Dressel, J. et al., 2014. Colloquium. *Reviews of Modern Physics* [Online], 86(1), pp.307–316. https://doi.org/10.1103/RevModPhys.86.307.

Duprey, Q., Kanjilal, S. et al., 2018. The Quantum Cheshire Cat Effect: Theoretical Basis and Observational Implications. *Annals of Physics* [Online], 391, pp.1–15. https://doi.org/10.1016/j.aop.2018.01.011.

Duprey, Q. and Matzkin, A., 2017. Null Weak Values and the Past of a Quantum Particle. *Physical Review A* [Online], 95(3), p.032110. https://doi.org/10.1103/PhysRevA.95.032110.

Duprey, Q. and Matzkin, A., 2018. Reply to “Comment on ‘null Weak Values and the Past of a Quantum Particle”. *Physical Review A* [Online], 97(4), p.046103. https://doi.org/10.1103/PhysRevA.97.046103.

Dziewior, J. et al., 2019. Universality of Local Weak Interactions and Its Application for Interferometric Alignment. *Proceedings of the National Academy of Sciences* [Online], 116(8), pp.2881–2890. https://doi.org/10.1073/pnas.1812970116.

Elitzur, A.C. and Vaidman, L., 1993. Quantum Mechanical Interaction-Free Measurements. *Foundations of Physics* [Online], 23(7), pp.987–997. https://doi.org/10.1007/BF00736012.

Englert, B.-G. et al., 2017. Past of a Quantum Particle Revisited. *Physical Review A* [Online], 96(2), p.022126. https://doi.org/10.1103/PhysRevA.96.022126.

Englert, B.-G. et al., 2019. Reply to "Comment on 'past of a Quantum Particle Revisited' ". *Physical Review A* [Online], 99(2), p.026104. https://doi.org/10.1103/PhysRevA.99.026104.

Gell-Mann, M. and Hartle, J.B., 1993. Classical Equations for Quantum Systems. *Physical Review D* [Online], 47(8), pp.3345–3382. https://doi.org/10.1103/PhysRevD.47.3345.

Gell-Mann, M. and Hartle, J.B., 1999. Innovation and Intellectual Property Rights. In: J. Fagerberg, D.C. Mowery and R.R. Nelson, eds. *Complexity, Entropy, and the Physics of Information*. Addison-Wesley, pp.266–290.

Geppert-Kleinrath, H. et al., 2018. Multifold Paths of Neutrons in the Three-Beam Interferometer Detected by a Tiny Energy Kick. *Physical Review A* [Online], 97(5), p.052111. https://doi.org/10.1103/PhysRevA.97.052111.

Griffiths, R.B., 1984. Consistent Histories and the Interpretation of Quantum Mechanics. *Journal of Statistical Physics* [Online], 36(1), pp.219–272. https://doi.org/10.1007/BF01015734.

Griffiths, R.B., 1996. Consistent Histories and Quantum Reasoning. *Physical Review A* [Online], 54(4), pp.2759–2774. https://doi.org/10.1103/PhysRevA.54.2759.

Griffiths, R.B., 2003. *Consistent Quantum Theory*. Cambridge: Cambridge University Press.

Griffiths, R.B., 2013. A Consistent Quantum Ontology. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* [Online], 44(2), pp.93–114. https://doi.org/10.1016/j.shpsb.2012.12.002.

Griffiths, R.B., 2014. The New Quantum Logic. *Foundations of Physics* [Online], 44(6), pp.610–640. https://doi.org/10.1007/s10701-014-9802-4.

Griffiths, R.B., 2015. Consistent Quantum Measurements. *Studies in History and Philosophy of Science Part B: Studies in History and Philosophy of Modern Physics* [Online], 52, pp.188–197. https://doi.org/10.1016/j.shpsb.2015.07.002.

Griffiths, R.B., 2016. Particle Path Through a Nested Mach-Zehnder Interferometer. *Physical Review A* [Online], 94(3), p.032115. https://doi.org/10.1103/PhysRevA.94.032115.

Griffiths, R.B., 2017. What Quantum Measurements Measure. *Physical Review A* [Online], 96(3), p.032110. https://doi.org/10.1103/PhysRevA.96.032110.

Hance, J.R., Ladyman, J. and Rarity, J., 2021. How Quantum Is Quantum Counterfactual Communication? *Foundations of Physics* [Online], 51(1), p.12. https://doi.org/10.1007/s10701-021-00412-5.

Hance, J.R., Rarity, J. and Ladyman, J., 2021. *Does the Weak Trace Show the Past of a Quantum Particle in an Unperturbed System?* [Online]. Available at: <https://arxiv.org/abs/2109.14060>.

Hashmi, F.A., Li, F., Zhu, S.-Y. and Zubairy, M.S., 2016. Two-State Vector Formalism and Quantum Interference. *Journal of Physics A: Mathematical and Theoretical* [Online], 49(34), p.345302. https://doi.org/10.1088/1751-8113/49/34/345302.

Hashmi, F.A., Li, F., Zhu, S.-Y. and Zubairy, M.S., 2018. Reply to the Comment on 'two-State Vector Formalism and Quantum Interference'. *Journal of Physics A: Mathematical and Theoretical* [Online], 51(6), p.068001. https://doi.org/10.1088/1751-8121/aa8d23.

Kamaruddin, S., Shaari, J.S. and Kolenderski, P., 2020. Counterfactual Protocol Within Device Independent Framework and Its Insecurity. *Scientific Reports* [Online], 10(1), p.5847. https://doi.org/10.1038/s41598-020-62812-3.

Lewis, D., 2001. *Counterfactuals*. 2nd ed. Oxford: Blakwell.

Li, Z.-H., Al-Amri, M. and Zubairy, M.S., 2013. Comment on "Past of a Quantum Particle". *Physical Review A* [Online], 88(4), p.046102. https://doi.org/10.1103/PhysRevA.88.046102.

Matzkin, A., 2019. Weak Values and Quantum Properties. *Foundations of Physics* [Online], 49(3), pp.298–316. https://doi.org/10.1007/s10701-019-00245-3.

Mermin, N.D., 2013. *Annotated Interview with a Qbist in the Making* [Online]. Available at: <https://arxiv.org/abs/1301.6551> [visited on 9 September 2022].

Noh, T.-G., 2009. Counterfactual Quantum Cryptography. *Physical Review Letters* [Online], 103(23), p.230501. https://doi.org/10.1103/PhysRevLett.103.230501.

Omnès, R., 1988. Logical Reformulation of Quantum Mechanics. I. Foundations. *Journal of Statistical Physics* [Online], 53(3), pp.893–932. https://doi.org/10.1007/BF01014230.

Omnès, R., 1994. *The Interpretation of Quantum Mechanics*. Princeton: Princeton University Press.

Peleg, U. and Vaidman, L., 2019. Comment on "Past of a Quantum Particle Revisited". *Physical Review A* [Online], 99(2), p.026103. https://doi.org/10.1103/PhysRevA.99.026103.

Potoček, V. and Ferenczi, G., 2015. Which-Way Information in a Nested Mach-Zehnder Interferometer. *Physical Review A* [Online], 92(2), p.023829. https://doi.org/10.1103/PhysRevA.92.023829.

Rao, V.N. and Srikanth, R., 2021. Noiseless Attack and Counterfactual Security of Quantum Key Distribution. *Physical Review A* [Online], 104(2), p.022424. https://doi.org/10.1103/PhysRevA.104.022424.

Rebufello, E. et al., 2021. Anomalous Weak Values Via a Single Photon Detection. *Light: Science & Applications* [Online], 10(1), p.106. https://doi.org/10.1038/s41377-021-00539-0.

Richter, M., Dziewit, B. and Dajka, J., 2018. The quantum cheshire cat effect in the presence of decoherence. *Advances in Mathematical Physics* [Online], 2018(1), p.7060586. https://doi.org/10.1155/2018/7060586.

Riedel, C.J., Zurek, W.H. and Zwolak, M., 2016. Objective Past of a Quantum Universe: Redundant Records of Consistent Histories. *Physical Review A* [Online], 93(3), p.032126. https://doi.org/10.1103/PhysRevA.93.032126.

Saldanha, P.L., 2014. Interpreting a Nested Mach-Zehnder Interferometer with Classical Optics. *Physical Review A* [Online], 89(3), p.033825. https://doi.org/10.1103/PhysRevA.89.033825.

Salih, H., 2015. Commentary: "Asking Photons Where They Have Been" - Without Telling Them What to Say. *Frontiers in Physics* [Online], 3, p.47. https://doi.org/10.3389/fphy.2015.00047.

Schlosshauer, M., 2007. *Decoherence and the Quantum-to-Classical Transition* [Online], *Frontiers Collection*. Berlin, Heidelberg: Springer. https://doi.org/10.1007/978-3-540-35775-9.

Shikano, Y. and Hosoya, A., 2009. Weak Values with Decoherence. *Journal of Physics A: Mathematical and Theoretical* [Online], 43(2), p.025304. https://doi.org/10.1088/1751-8113/43/2/025304.

Sokolovski, D., 2018. Comment on "Null Weak Values and the Past of a Quantum Particle". *Physical Review A* [Online], 97(4), p.046102. https://doi.org/10.1103/PhysRevA.97.046102.

Sponar, S. et al., 2019. Asking Neutrons Where They Have Been. *Journal of Physics: Conference Series* [Online], 1316, p.012002. https://doi.org/10.1088/1742-6596/1316/1/012002.

Vaidman, L., 1999. Time-Symmetrized Counterfactuals in Quantum Theory. *Foundations of Physics* [Online], 29(5), pp.755–765. https://doi.org/10.1023/A:1018826607583.

Vaidman, L., 2013a. Past of a Quantum Particle. *Physical Review A* [Online], 87(5), p.052104. https://doi.org/10.1103/PhysRevA.87.052104.

Vaidman, L., 2013b. Reply to "Comment on 'past of a Quantum Particle'". *Physical Review A* [Online], 88(4), p.046103. https://doi.org/10.1103/PhysRevA.88.046103.

Vaidman, L., 2014. Tracing the Past of a Quantum Particle. *Physical Review A* [Online], 89(2), p.024102. https://doi.org/10.1103/PhysRevA.89.024102.

Vaidman, L., 2016a. Comment on "One-State Vector Formalism for the Evolution of a Quantum State Through Nested Mach-Zehnder Interferometers". *Physical Review A* [Online], 93(3), p.036103. https://doi.org/10.1103/PhysRevA.93.036103.

Vaidman, L., 2016b. Comment on "Which-Way Information in a Nested Mach-Zehnder Interferometer". *Physical Review A* [Online], 93(1), p.017801. https://doi.org/10.1103/PhysRevA.93.017801.

Vaidman, L., 2017a. Comment on "Particle Path Through a Nested Mach-Zehnder Interferometer". *Physical Review A* [Online], 95(6), p.066101. https://doi.org/10.1103/PhysRevA.95.066101.

Vaidman, L., 2017b. Weak Value Controversy. *Philosophical Transactions of the Royal Society A: Mathematical, Physical and Engineering Sciences* [Online], 375(2106), p.20160395. https://doi.org/10.1098/rsta.2016.0395.

Vaidman, L., 2018. Comment on 'two-State Vector Formalism and Quantum Interference'. *Journal of Physics A: Mathematical and Theoretical* [Online], 51(6), p.068002. https://doi.org/10.1088/1751-8121/aa8d24.

Vaidman, L., 2019. Analysis of Counterfactuality of Counterfactual Communication Protocols. *Physical Review A* [Online], 99(5), p.052127. https://doi.org/10.1103/PhysRevA.99.052127.

Vaidman, L., 2020. Neutrons and Photons Inside a Nested Mach-Zehnder Interferometer. *Physical Review A* [Online], 101(5), p.052119. https://doi.org/10.1103/PhysRevA.101.052119.

Vaidman, L., Ben-Israel, A. et al., 2017. Weak Value Beyond Conditional Expectation Value of the Pointer Readings. *Physical Review A* [Online], 96(3), p.032114. https://doi.org/10.1103/PhysRevA.96.032114.

Vaidman, L., Danan, A., Farfurnik, D. and Bar-Ad, S., 2015. Response: Commentary: "Asking Photons Where They Have Been" - Without Telling Them What to Say. *Frontiers in Physics* [Online], 3, p.48. https://doi.org/10.3389/fphy.2015.00048.

Wander, A., Cohen, E. and Vaidman, L., 2021. Three Approaches for Analyzing the Counterfactuality of Counterfactual Protocols. *Physical Review A* [Online], 104(1), p.012610. https://doi.org/10.1103/PhysRevA.104.012610.

Wieśniak, M., 2018. Spectra in Nested Mach–Zehnder Interferometer Experiments. *Physics Letters A* [Online], 382(36), pp.2565–2568. https://doi.org/10.1016/j.physleta.2018.06.032.

Zhou, Z.-Q. et al., 2017. Experimental Observation of Anomalous Trajectories of Single Photons. *Physical Review A* [Online], 95(4), p.042121. https://doi.org/10.1103/PhysRevA.95.042121.