Creating new concepts in mathematics: freedom and limitations. The case of Category Theory

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Abstract

In the paper we discuss the problem of limitations of freedom in mathematics and search for criteria which would differentiate the new concepts stemming from the historical ones from the new concepts that have opened unexpected ways of thinking and reasoning.

We also investigate the emergence of category theory (CT) and its origins. In particular we explore the origins of the term *functor* and present the strong evidence that Eilenberg and Carnap could have learned the term from Kotarbiński and Tarski.

Keywords

categories, functors, Eilenberg-Mac Lane Program, mathematical cognitive transgressions, phylogeny, platonism.

Philosophical Problems in Science (Zagadnienia Filozoficzne w Nauce)
No 69 (2020), pp. 33–65

1. Introduction

The celebrated dictum of Georg Cantor that "The *essence* of *mathematics* lies precisely in its *freedom*" expressed the idea that in mathematics one can freely introduce new notions (which may, however, be abandoned if found unfruitful or inconvenient).[1] This way Cantor declared his opposition to claims of Leopold Kronecker who objected to the free introduction of new notions (particularly those related to the infinite).

Some years earlier Richard Dedekind stated that—by forming, in his theory, a cut for an irrational number—we *create* a new number. For him this was an example of a constructed notion which was a free creation of the human mind (Dedekind, 1872, § 4).

In 1910 Jan Łukasiewicz distinguished *constructive notions* from empirical *reconstructive* ones. He referred (with reservation) to Dedekind's statement and pointed out that a consequence of our "creation" of those notions is the spontaneous emergence of countless relations which no more depend on our will (Łukasiewicz, 1910).

Until the discovery of non-Euclidean geometries, geometry was regarded as an abstraction of the spatial reality. The freedom of creation in geometry was limited by this reality. Hilbert advocated a formal point of view, broadening the freedom of choosing the axioms, while Poincaré maintained that the axioms of geometry are conventions.

Clearly, the freedom of mathematics is limited by logical constraints. At the same time, logical inference yields deep meaning to mathematics. As Michał Heller put it, "If I accept one sentence, I must also accept another sentence. Why must I? Who forces me? Nobody. Yet, I must. Generally, we bear badly any restrictions of our liberty, but in the case of mathematical deduction inevitability of the

[1] The italics in the original sentence "Das *Wesen der Mathematik* liegt gerade in ihrer *Freiheit*" are Cantor's. The first version was published in 1879, reprinted in (Cantor, 1883, p.34), discussed by Ferreirós (1999, p.257).

conclusion gives us the feeling of safety (I have not deviated from the way) and of the accompanying intellectual comfort, sometimes even great joy" (Heller, 2015, p.21).

A mathematician trying to prove a theorem knows the feeling of an invisible wall which blocks some intended arguments. Also new concepts must be consistent with earlier ones and must not lead to contradiction or ambiguity. Moreover, in mathematical practice, only intersubjective mental constructions are accepted.

The purpose of this paper[2] is to look for restraints and patterns in the historical development of mathematics.[3] Some types of paths will be distinguished, first generally, and then they will be used to highlight some features of the rise of category theory (CT).

Michael Atiyah, in his *Fields Lecture* at the World Mathematical Year 2000 in Toronto, expressed his view that

> it is very hard to put oneself back in the position of what it was like in 1900 to be a mathematician, because so much of the mathematics of the last century has been absorbed by our culture, by us. It is very hard to imagine a time when people did not think in those terms. In fact, if you make a really important discovery in mathematics, you will then get omitted altogether! You simply get absorbed into the background. (Atiyah, 2002, p.1)

This statement may appear startling, as the mathematical meaning of a text from around 1900 is generally believed to be time-proof. Yet what Atiyah had in mind was not the meaning of published texts—definitions, theorems and proofs— but the way mathematicians thought at that time.

[2] The present paper is based on a talk delivered at XXIII Kraków Methodological Conference 2019: *Is Logic a Physical Variable?*, 7-9 November 2019.

[3] The significant question of degree of freedom in mathematical conceptualization of physical reality is not considered here.

It was 75 years ago when the celebrated paper by Eilenberg and Mac Lane was published.[4] This event marked the rise of category theory (CT). As the development of mathematics accelerated in the second half of the 20th century, it may be hard to fully imagine how mathematicians thought in 1945. Their definitions, theorems and comments are clear, but one should be aware that a reconstruction of their ideas, their thinking may be specifically biased by our present understanding of the mathematical concepts involved.

2. Background conceptions

The main ideas of this paper—which includes a very wide spectrum of examples, from ancient Greek mathematics to modern, from children's counting to CT—are the following:

- A mathematical concept, no matter how novel, is never independent of the previous knowledge; it is based on a reorganization of existing ideas.
- A radically new mathematical idea never germinate in somebody's mind without a period of incubation, usually a lengthy one.
- There is a long distance to cover between a spontaneous, unconscious use of a mathematical idea or structure in a concrete setting, and a conscious, systematic use of it (Piaget and García, 1989, p.25).
- A person who has achieved a higher level of mathematical thinking is often unable to imagine thinking of a person from

[4] Mac Lane earlier in his life, in particular in (Eilenberg and Mac Lane, 1945) and (Mac Lane, 1950) wrote his name as MacLane. Later he began inserting a space into his surname, in particular in (Mac Lane, 1971).

another epoch or of a present learner and, consequently, may unconsciously attribute to him/her an inappropriate (too high or too low) level of thinking.

Transgressions

A *mathematical cognitive transgression* (or briefly: a *transgression*) is defined as crossing—by an individual or by a scientific community—of a previously non-traversable limit of own mathematical knowledge or of a previous barrier of deep-rooted convictions (Semadeni, 2015). Moreover, it is assumed that:

1. the crossing concerns a (broadly understood) mathematical idea and the difficulty is inherent in the idea,
2. the crossing is critical to the development of the idea and related concepts,
3. it is a passage from a specified lower level to a new specified upper level,
4. the crossing is a result of conscious activity (the activity need not be intentional and purposely orientated towards such crossing; generally such effect is not anticipated in advance and may even be a surprise).

In the history of mathematics there were numerous transgressions of different importance, some great ones and many "mini-transgressions". Usually they were not single acts—they involved a global change of thinking which matured for years or even generations and they were based on the work of many people. In ancient times two transgressions were the most significant:

- The transition from practical dealing with specific geometric shapes to deductive geometry.
- The celebrated discovery (in the 5^{th} century BC) that the diagonals of a square are incommensurable with their sides, that they are ἄλογος (*a-logos*), without a ratio, irrational (in modern setting, foreign to Greek thinking, it was the irrationality of $\sqrt{2}$) (Baszmakowa, 1975, pp.80-81). The Pythagorean paradigm was undermined by this απορία (*aporia*). Their understanding of mathematics was eroded. However, nothing certain is known about this discovery. Stories presented in popular books are based on doubtful legends from sources written seven or eight centuries later (Knorr, 1975, p.21, 51). This incommensurability could not be a single discovery by an individual. It must have been a lengthy process. Never in the historical development of mathematics such a major change occurred in a short time.

In modern history there were many transgressions. Let us list some of the best known:

- Acceptance of negative numbers.
- The transition from potential infinity (infinity at a *process* level) to the *actual* infinity (infinity as an object).
- The emergence of projective geometry.
- The discovery of non-Euclidean geometries.

We will discuss the case of CT, arguing in particular that creating the theory of *elementary topoi* in CT should be regarded as a major transgression.

Phylogeny and ontogeny

The term *phylogeny* refers here to the evolutionary history of mathematics (or rather to its modern reconstructions), from ancient times on. *Ontogeny* denotes the development of basic mathematical concepts and structures in the mind of an individual person, from early childhood. Phylogeny and ontogeny are in some sense complementary descriptions (Freudenthal, 1984; Piaget and García, 1989, pp.4-29).

In case of mathematics, the oft-quoted phrase: *ontogeny recapitulates phylogeny* implies that one can learn from the history of old mathematics for the sake of present teaching. The history of mathematics sometimes gives useful hints, e.g. one may argue that since the historical process of forming the general concept of a function took centuries (from Descartes, if not much earlier, to Peano and Hausdorff), we should not expect that a secondary school student can grasp it—learning Dirichlet's description—after a few lessons. The general concept of the function was not yet quite clear to mathematicians of the first half of the 19$^{\text{th}}$ century (Lakatos, 1976, Appendix 2; Youschkevitch, 1976; Ferreirós, 1999, pp.27–30).

On the other hand, the idea that ontogeny recapitulates phylogeny may be misleading. Piaget always stressed that arithmetic cognition results from logico-mathematical experience with concrete objects, pebbles say, and is educed from *the child's actions* rather than from heard words. It is abstracted from a coordination of intentional motions and accompanying thoughts. Nevertheless, Piaget was in favour of the phylogeny-ontogeny parallelism and reasoned roughly as follows. Since one-to-one correspondence preceded numerical verbal counting in the very early periods of human civilization (evidenced by artefacts such as notched bones and also found in crude unlettered tribes), the same should apply to children. Cantor's theory of cardinal numbers confirmed this thinking (Beth and Piaget, 1966, pp.259–260). Consequently, Piaget and many educators insisted on one-to-one correspondence as a foundation of early school arithmetic, neglecting the fact that nowadays children learn number names early, often together with learning to speak, and moreover counting is now deeply rooted culturally. Research evidence shows that counting, rather than one-to-one correspondence, is a basis of the child's concept of number (Gelman and Gallistel, 1978, pp.77–82). In this way the phylogeny–ontogeny parallelism adversely affected early mathematics education in the time of the 'New Math' movement.

Hans Freudenthal, in the context of mathematics, suggested the converse idea: *What can we learn from educating the youth for understanding the past of mankind?* This reverses the traditional direction of inference in the phylogeny–ontogeny parallelism. In particular, one may ask whether contemporary knowledge of the difficulties in the transition from the concrete to more abstract mathematical reasoning of children may be helpful in better understanding of limitations of our reconstructions of the development of the early Greek mathematics.

In the sequel, certain aspects of the development will be traced both in phylogeny and in ontogeny, inextricably intertwined with the the mathematical questions themselves.

Platonizing constructivism in mathematics

The theoretical framework of the paper is platonizing constructivism in mathematics (Semadeni, 2018). It is assumed that:

- each of the three major positions in the philosophy of mathematics from the beginning of the 20^{th} century: *platonism*, *constructivism*, *formalism* describes some inherent, complementary features of mathematics;
- these three major positions can be reconciled provided that they are regarded as *descriptive*, as an account of some inherent features of mathematics, and *not normative*, i.e., when one leaves out the eliminating words (as 'only', 'oppose') which explicitly deny other standpoints. Moreover, various versions of the three positions often overlap.

In the sequel, the term *constructivism* will not be understood as in papers on foundations of mathematics, but rather in a way akin to its meaning in research on mathematics education, related to post-Piagetian psychological versions of constructivism. Briefly, one assumes here that humans construct mathematical concepts in their minds and discover their properties. A concept develops its necessary structure as a consequence of its context and—in the long term—becomes *cristalline* in the sense of David Tall (2013, p.27); then its properties appear independent of our will. This phenomenon may be traced both in phylogeny and in ontogeny. Moreover, in each essential progression, new mental structures are build on the preceding ones and are always integrated with previous ones (Piaget and García, 1989, pp.22–29).

By platonizing constructivism we mean an analysis—in constructivistic terms—of sources and consequences of the platonistic attitude of a majority of mathematicians and contrasting them with the well-known difficulties of consistent platonism in the philosophy of mathematics.

3. Developmental successors

After the introductory examples we now look for ways to distinguish between:

- mathematical concepts which—historically—were natural successors to previous ones,
- mathematical concepts which could be conceived and defined only after opening new paths of thought and reasoning.

The following metaphorical labels will be used: *onward development*, *branching-off*, *upward development*, *downward development*. These labels will be interpreted with examples. We do not expect to find clear criteria, but the ensuing discussion may be illuminating.

The *emergence of numerals* in the Late Stone Age is evidenced by tally marks (in the form of notched bones). Ethnologists have found that early tribes had only two counting words: *one* and *two*, followed by *many*. Also in present Indo-European languages these two numbers and their ordinal counterparts are linguistically different from the following numbers. It has been suggested that the proto-Indo-European number **trei* (three) was derived from the verb **terh* (meaning: *pass*); thus, the word *three* is related to *trans*. This may be a hint of a very ancient mental obstacle between numbers *two* and *three*.

One may conjecture that *after the passage from 2 to 3 there was no notable obstacle to gradual development of unlimited counting*. Of course, the actual development took centuries, if not millennia. Anyway, for present children there is no hurdle between 2 and 3, as they are taught counting very early. Moreover, counting starts to make sense with three items.

Onward development

Onward development of indefinite counting includes its *developmental successors*: simple addition of natural numbers (which develops through a stage called *count all* and then a more advanced stage *count on*), subtraction (as taking away), multiplication, division (originally there are two kinds of it: *equal sharing* and *equal grouping*), and even simple powers, all within some range of natural numbers.

These concepts are included in the onward development of counting, by virtue of the following features:

- *no branching*: each new concept naturally comes after the previous ones;
- *ontological stability*: each concept (e.g., number 17 or the product 3×6), remains essentially the same object, although the related ideas are enriched after each extension of the scope of arithmetic and—in the historical development—are subject to evolutionary changes.

The conception of developmental successors, outlined here, does not take into account a relative difficulty of concepts; what is crucial is whether these concepts follow the previous lines of thinking.

Branching-off

The concept of "branching-off" arises from a negation of the first requirement in the description of an onward development. An example of it are fractions, which *branch off* from natural numbers; it is not onward development, although there are many ties between natural numbers and fractions.

Fractions could be introduced to children in two ways. In the first, some idealized whole is divided into m equal parts and then n of them are taken. In the second, n whole things are equally divided into m parts. They are two main aspects of the concept of a fraction. For instance, $\frac{3}{4}$ of pizza may be obtained by cutting it into 4 parts and taking 3 such parts (thus $\frac{3}{4} = 3 \times \frac{1}{4}$). A more advanced way of thinking of $\frac{3}{4}$ is 3 divided by 4; the latter may be explained with the example of 3 pizzas to be divided among 4 persons.

The distinction looks quite elementary. Yet, it was significant in the phylogeny of fractions. First procedure is akin to that of ancient Egyptians, the second—to Greek ratios; both were inherited by Arabic mathematicians. In the ontogeny the two ways are always present, but not necessarily noticed. The following reminiscence by William Thurston (1946-2012), written 8 years after he had received Fields Medal, describes his discovery of the identification of previously different objects.

> I remember as a child, in fifth grade, coming to the amazing (to me) realization that the answer to 134 divided by 29 is $134/29$ (and so forth). What a tremendous labor-saving device! To me, '134 divided by 29' meant a certain tedious chore, while $134/29$ was an object with no implicit work. I went excitedly to my father to explain my major discovery. He told me that of course this is so, a/b and a divided by b are just synonyms. To him it was just a small variation in notation (Thurston, 1990, p.848).

The fraction $\frac{a}{b}$ becomes identified with the result of division $a \div b$ and—in this synthesis—they both form a single mathematical object. Philosophically, however, it is an ontological change: two different beings, results of two different mental constructions, become regarded as a single one.

Onward branching-off can be traced in many parts of mathematics. Calculus branches off from a theory of the field of real numbers (axiomatic or based on a construction). Infinite sequences of real numbers branch off from elementary algebra of real numbers. Limits of sequences branch off from general theory of sequences. These examples vividly show that the question of distinguishing branching-off from onward development is delicate, as the criteria are far from being precise, but it may contribute to better understanding the historical development of mathematics.

Upward development and downward development

By *upward development* of a piece of mathematics we mean passing from some concepts and relations between them to a more abstract version of them. Examples:

- Transition from practical addition (verbalized as, e.g., *two and three make five*) to symbolic version (e.g., $2 + 3 = 5$) took centuries (the sign $+$ appeared in some 15$^{\text{th}}$ century records; the first occurrence of the equality sign $=$ was found in a text by a Welsh mathematician Robert Recorde from 1557).
- Transition from the space $\mathbb{R}^n$ to an axiomatically given vector space over $\mathbb{R}$.
- Transition from a vector space over $\mathbb{R}$ to a vector space over a field.
- Transition from group theory to the category **Grp**.
- Transition from a category to a metacategory (in the sense of (Mac Lane, 1971, pp.7–11)).

Downward development is—in some sense—an inverse process, to more concrete questions or to a lower level of abstraction, so the above examples may be used the other way round. Also some typical applications of mathematics may be included here, e.g., the passage from abstract Boolean algebras to a description of certain types of electrical circuits, as conceived by Claude Shannon (1936).

In the 20th century the mathematics grew rapidly and the upward development became much easier mentally as a result of both: a general change of the attitude of mathematicians toward abstraction and the routine of expressing all concepts in the language of set theory. Branching-off were so frequent that the above metaphors are of little use. There is, however, a notable exception: a new theory which opened a new direction of thinking, so its beginnings may be discussed in a way akin to that used with respect to distant past.

4. The rise of category theory (CT)

A very special feature of CT is that it has a pretty precise date of its official birth: the publication of the paper by Samuel Eilenberg and Saunders Mac Lane (1945). It was presented at a meeting of the American Mathematical Society in 1942 and published in 1945.

According to the Stanford Encyclopedia of Philosophy, *CT "appeared almost out of nowhere"*. Not quite so. As in any mathematical theory, some CT ideas had been conceived much earlier, particularly in algebraic topology, and some of them can be traced to the 19th century.

Many conceptual transformations—either explicit and well recognized or used implicitly, without awareness—contributed to the rise of CT. Going back, a significant factor was the historic development of the mathematical concept of a function.

Until the beginning of the 19$^{\text{th}}$ century, a general symbol for a function (f or φ) was almost non-existent (Youschkevitch, 1976). A significant step toward CT was the general notion of a mapping introduced by Dedekind in 1888. In his *Enklärung* (*explanation*) he did not define the concept of *Abbildung* φ (literally: *image* or *representation*) from a set (*System*) S into a set S', but interpreted it generally as an arbitrary law (*Gesetz*) according to which to each element s there corresponds (*gehört*) a certain thing (*Ding*) $S' = \varphi(s)$, called the image (*Bild*) of s. He also defined a composed mapping (*zusammengesetzte Abbildung*) $\vartheta(s) = \psi(s') = \psi(\varphi(s))$ of two given ones, denoted as $\varphi.\psi$ or $\varphi\psi$, defined injective mappings (*ähnlich* or *deutlich*), the inverse mapping and proved their main properties (Dedekind, 1888, §2–4; Ferreirós, 1999, p.88–90, 228–229).

Emmy Noether in her lectures in the 1920s emphasised the role of homomorphisms in group theory. Before her, groups were understood as generators and relations (in modern terms, as quotients of free groups). She also argued that the homology of a space is a group, is an algebraic system rather than a set of numbers assigned to the space. Her lectures and the lectures of Emil Artin formed a basis for the celebrated book by van der Waerden (1930) on modern algebra. This current of thought led to CT.

Generally, in the symbol of the type $f(x)$, the part f was always understood as fixed and x was a variable. At the end of the 1920s, however, in functional analysis and related fields, a new way of thinking emerged. In certain situations the roles of symbols in $f(x)$ reversed: the point x was regarded as fixed while the function f became a variable (as an element of a function space), e.g., $x \in [0,1]$ was fixed and f was a variable in the space $C([0,1])$ of continuous functions on the interval $[0,1]$. In this new role, the point x became a functional δ_x on $C([0,1])$. Such a change of the roles function-element became crucial, e.g., in the Potryagin duality of locally compact abelian groups (Hewitt and Ross, 1963) and in Gelfand's theory of commutative Banach algebras. It was also used by Eilenberg and Mac Lane in their first example (finite dimensional vector spaces and their dual spaces) motivating the concept of a natural equivalence.

A crucial example of a contravariant functor was the adjoint T^* of a linear operator T on a Hilbert space, introduced in 1932 by John von Neumann.[5]

According to Mac Lane, abstract algebra, lattice theory and universal algebra were necessary precursors for CT. However, he also suggested that certain notational devices preceded the definition of a category. One of them was the fundamental idea of representing a function by an arrow $f\colon X \to Y$, which first appeared in algebraic topology about 1940, probably introduced by Polish-born topologist Witold Hurewicz (Mac Lane, 1971, p.29; 1988, p.333). Originally, it looked as just another symbol, but from a later perspective the use of such symbol was one of the key changes. Thus, a notation (the arrow) led to a concept (category). Such new symbols got absorbed into the background of mathematical thinking, and was later used as something obvious. Together with commutative diagrams, which were probably also first used by Hurewicz, they paved the way to CT.

[5] Mac Lane (1988, p.330) tells a story of how Marshall Stone advised von Neumann to introduce the symbol T^* and how it changed the publication. He also mentions a fact which may interests philosophers: in 1929 von Neumann lectured in Göttingen and presented his axiomatic definition of a Hilbert space, while David Hilbert—listening to it—evidently thought of it as of the concrete space ℓ^2, not in the axiomatic setting.

Mac Lane often accented two features of mathematics: computational and conceptual. He noted that the initial discovery of CT came directly from a problem of calculation in algebraic topology (Mac Lane, 1988, p.333).

Eilenberg and Mac Lane were aware that they introduced very abstract mathematical tools, which did not fit any algebraic system in the Garrett Birkhoff's universal algebra. It might seem too abstract and was certainly off beat and a "far out" endeavour. Although it was carefully prepared, it might not have seen the light of day (Mac Lane, 2002, p.130).

The origin of the term *functor*

Mac Lane has written *Categories, functors, and natural transformations were discovered by Eilenberg–Mac Lane in 1942* (Mac Lane, 1971, p.29). The word "discovered" may be regarded as an indication of a hidden Platonistic attitude of Mac Lane, in spite of his verbal declarations against Platonism (Mac Lane, 1986, pp.447–449; Król, 2019; Skowron, manuscript). He also wrote:

> Now the discovery of ideas as general as these is chiefly the willingness to make a brash or speculative abstraction, in this case supported by the pleasure of purloining words from the philosophers: "Category" from Aristotle and Kant, "Functor" from Carnap (*Logische Syntax der Sprache*) (Mac Lane, 1971, pp.29–30).

This sentence has been taken very seriously by several authors. However, the way it was phrased suggests that it was rather intended to be a delicate joke.[6] Attributing the origin of the term *category* to Aristotle and Kant is clear, although in 1899 René-Louis Baire (in his *Thèse*) introduced—in another context—the word *category* to mathematics.[7] Concerning the origin of the term *functor* in CT, one may recall the following facts:[8]

- In Rudolf Carnap's book *Abriss der Logistik* (Carnap, 1929) the term "Funktor" does not appear.
- In 1929 the Polish term "funktor" was used in propositional calculus by Tadeusz Kotarbiński in his book (Kotarbiński, 1929).
- Alfred Tarski often emphasised that Kotarbiński had been his teacher (Feferman and Feferman, 2004, part 2).
- Carnap met Tarski in Vienna in February 1930 and visited Warsaw in November 1930; he learned much from Tarski.

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[6] Let us note that the noun *brash* means a mass of fragments; according to *Cambridge International Dictionary of English* (1995), the adjective *brash* is disapproving, referring to people who show too much confidence and too little respect, while *Webster's New World Dictionary* (1984) lists—as meanings of *brash*—also *hasty and reckless*, *offensively bold*. On the other hand, the word *purloining* means *stealing* or *borrowing without permission*. Such a comment (with the word *pleasure*) by Mac Lane concerning CT could not be serious. On the other hand, in 2002 Mac Lane came back to Carnap, adding: "Also the terminology was largely purloined: "category" from Kant, "natural" from vector spaces and "functor" from Carnap. (It was used in a different sense in Carnap's influential book *Logical Syntax of Language*; I had reviewed the English translation of the book (in the Bulletin AMS 1938) and had spotted some errors; since Carnap never acknowledged my finding, I did not mind using his terminology)" (Mac Lane, 2002, pp.130–131).

[7] A subset A of a topological space X is called *a set of first category* (*un ensemble de première catégorie*) in X iff A is the union of a countable family of nowhere dense sets; otherwise it is *a set of the second category* (*Menge erster und zweiter Kategorie*). The celebrated *Baire category theorem* states, in a generalized form, that a complete metric space is not a set of the first category (Hausdorff, 1914, p.328; Kuratowski, 1933, § 10). The clumsy term *set of the first category* was later replaced by the term a *meager set* (Kelley, 1955, p.201). In the 1930s Baire category theorem was a very popular tool in the Warsaw school of topology, so Eilenberg must have known it.

[8] The author is indebted to Professor Jan Woleński for the relevant information.

- In 1933 Tarski, in the Polish version of his famous paper *On the concept of truth in formal languages* Tarski (1933) used the term "funktor" and mentioned that he owed the term to Kotarbiński.
- Carnap used the term "Funktor" in his book (1934) (quoted by Mac Lane) in a sense more general than that of Kotarbiński and Tarski.
- Eilenberg studied mathematics in Warsaw from 1930. In 1931 he attended Tarski's lectures on logic (Feferman and Feferman, 2004, part 3 and 12). He left Warsaw in 1939.

This evidence strongly suggests that both Carnap and Eilenberg could have learned, independently, the term from Kotarbiński and Tarski.

Was the original CT an onward development of previous mathematical theories?

Using a metaphor explained above, one may argue that the definition of a category and of a functor were within a major onward development of part of mathematics of the first half of the 20$^{\text{th}}$ century, that is set theory, algebra, topology etc. Indeed, for a person working in group theory, say, a natural continuation should be to think of all groups, their homomorphisms, isomorphisms, and the composites as of a single whole: *group theory*. Similarly one could think of vector spaces with linear maps as of another whole. Some analogies between theories were obvious. Moreover, the axioms of CT are reminiscent of those of semigroup theory. The concept of a covariant functor was a natural analogue of homomorphisms of algebras. Contravariant functors had been present in various duality theories (e.g., in Pontryagin's duality mentioned above). CT provided general concepts applicable to all branches of abstract mathematics, contributed to the trend towards uniform treatment of different mathematical disciplines, provided opportunities for the comparison of constructions and of isomorphisms occurring in different branches of mathematics, and may occasionally suggest new results by analogy (Eilenberg and Mac Lane, 1945, p.236).

The great achievement of Eilenberg and Mac Lane was the idea that a formalization of various evident analogies was worth systematizing and publishing. The initial neglect of the paper of Eilenberg and Mac Lane (1945) by mathematicians was very likely a result of the fact that it was regarded as a long paper within onward development of known part of mathematics, with many rather simple definitions and examples, tedious verification of easy facts, and no theorem with an involved proof. Ralf Krömer, in his book on the history and philosophy of CT, has outright stated that Eilenberg and Mac Lane needed to have remarkable courage to write and submit for publication the paper almost completely concerned with conceptual clarification (Krömer, 2007, p.65).

A novelty of (Eilenberg and Mac Lane, 1945, p.272), which at first appeared insignificant, was regarding elements p_1, p_2 of a single quasi-ordered set P as objects of a category, with a unique morphism $p_1 \to p_2$ iff $p_1 \leq p_2$ and no morphisms otherwise. This opened a way to a series of generalizations, in particular regarding certain commutative diagrams as functors on small categories.

One may argue that this achievement was still within onward development of CT as it was within the scope of previous knowledge. Let us recall that the difficulty and the originality of a theorem are not taken into account; what is crucial is whether the concepts involved are natural extension of the previous knowledge and thinking.

The category axioms represent a very weak abstraction (Goldblatt, 1984, p.25). In spite of this fact, a few years later the conceptual clarification turned out to be highly effective in the book *Foundations of Algebraic Topology* written by Eilenberg together with Norman Steenrod (1952). The latter admitted in a conversation that the 1945 paper on categories had a more significant impact on him than any other research paper, it changed his way of thinking.

The Eilenberg-Mac Lane Program

This program has been formulated as follows:

> The theory [. . .] emphasizes that, whenever new abstract objects are constructed in a specified way out of given ones, it is advisable to regard the construction of the corresponding induced mappings on these new objects as an integral part of their definition. The pursuit of this program entails a simultaneous consideration of objects and their mappings (in our terminology, this means the consideration not of individual objects but of categories). [. . .]
>
> The invariant character of a mathematical discipline can be formulated in these terms. Thus, in group theory all the basic constructions can be regarded as the definitions of co- or contravariant functors, so we may formulate the dictum: The subject of group theory is essentially the study of those constructions of groups which behave in a covariant or contravariant manner under induced homomorphisms. More precisely, group theory studies functors defined on well specified categories of groups, with values in another such category. This may be regarded as a continuation of the Klein Erlanger Programm, in the sense that a geometrical space with its group of transformations is generalized to a category with its algebra of mappings.

> [...] such examples as the "category of *all* sets", the "category of *all* groups" are illegitimate. The difficulties and antinomies are exactly those of ordinary intuitive *Mengenlehre*; no essentially new paradoxes are involved. [...] we have chosen to adopt the intuitive standpoint, leaving the reader free to insert whatever type of logical foundation (or absence thereof) he may prefer. [...]
>
> It should be observed first that the whole concept of a category is essentially an auxiliary one; our basic concepts are essentially those of a functor and of a natural transformation. [...] The idea of a category is required only by the precept that every function should have a definite class as domain and a definite class as range, for the categories are provided as the domains and ranges of functors. (Eilenberg and Mac Lane, 1945, p.236–237, 246–247)

The quoted comparison of CT to the celebrated program of Felix Klein shows vividly that the authors regarded their work as significant. An important novelty of (Eilenberg and Mac Lane, 1945) was to use the same letter to denote both: the object component of a functor and its morphism component. This had not been a common practice, even when these correspondences were dealt with in a single paper. This novelty and the whole program fit well Atiyah's conception (quoted above) of ideas *absorbed by mathematicians' culture*.

CT is both a specific domain of mathematics and at the same time a conceptual framework for a major part of modern theories.

5. The amazing phenomenon of unexpected branchings-off in CT

Up to this point CT might be regarded as being within onward development of earlier theories: algebra, topology, functional analysis.

However, a branching-off in (Eilenberg and Mac Lane, 1945) is the concept of a *natural equivalence* (central in the title of the paper), with an essential use of commutative diagrams.[9] It was a completely new idea, but its initial impact was limited.

After 1945 CT—as a general theory—lay dormant till the emergence of significant new concepts and a breaking series of major branchings-off in the second half of the 1950s. One of their outstanding features was a new type of a definition, formulated in the form of a *unique factorization problem*.

First such explicit definition appeared in the paper by Pierre Samuel (1948), a member of the Bourbaki group, on free topological groups, albeit it was still in the language of set theory, without arrows. Two years later commutative diagrams were demonstrated as a convenient tool in such problems by Mac Lane (1950). The concept of a *dual category*, formulated by Eilenberg and Mac Lane in (1945, p.259) and further developed by Mac Lane (1950), had its conceptual roots in various duality theories, particularly in that of projective geometry.

The *product* of two categories was an analogue of that for groups and various algebras. Mac Lane analysed the concept of duality, stressed diagrammatic dualities of various pairs of concepts and presented the definitions of *direct* and *free products* in group theory (later generalized to the concepts of categorial *products* and *coproducts*, respectively).

[9] It is not clear why Eilenberg and Mac Lane refrained from setting the concept in the general form of a *natural transformation* (examples abounded). Perhaps they felt they should not pursue a still more general setting without accompanying results.

A metamorphosis from Eilenberg–Mac Lane Program to mature CT

A turning point in the development of CT was the seminal paper by Daniel Kan (1958) on *adjoint functors*. In much the same time period, independently, several closely related concepts and results were worked out: *representable functors*, *universal morphisms*, *Yoneda lemma*, various types of *limits* and *colimits* (Mac Lane, 1988, pp.345–352). Special cases of them had a long earlier history in specific situations in algebra and topology (e.g., Freudenthal's theorems on *loops* and *suspensions* in homotopy theory proved in 1937). This confirms a known phenomenon that mathematicians may use an idea spontaneously, without being conscious of it in a more abstract setting.

Mac Lane (1950) also opened the way to the study of categories with additional structure, which some years later developed to the study of abelian categories. This topic was developed—in a remarkably short time—due to the work of Alexandre Grothendieck, David Buchsbaum, Pierre Gabriel, Max Kelly and authors of two monographs: Peter Freyd (1964) and Barry Mitchell (1965).

Within 20 years CT, originally conceived as a useful language for mathematicians, became a developed, mature theory, something totally unexpected by its founders.

Set theory without elements

The results of the work of William Lawvere turned out to be not only a new branch of CT, but also opened new perspectives in mathematics, logic, foundations of mathematics, and philosophy. In his Ph.D. thesis at Columbia University in New York, supervised by

Eilenberg (defended in 1963, known from various copies, with full text published 40 years later) many new ideas were presented, including a categorical approach to algebraic theories (Lawvere, 1963).

Lawvere also tackled the general question as to what conditions a category must satisfy in order to be equivalent to the category **Set**. The idea looked analogous to the so-called *representation theorems*, i.e., propositions asserting that any model of the axioms for a certain abstract structure must be (in some prescribed sense) isomorphic to a specific type of models of the theory or to one particular concrete model.[10] However, Lawvere's case was unique and controversial in the sense that his 'sets' were conceived *without elements*. The theory did not have the primitive notion "element of". And it did work.

Specifically, Lawvere characterized **Set** (up to *equivalence* of categories) as a category $\mathcal{C}$ with the following: an *initial* object **0**; a *terminal* object **1** (which gives rise to *elements* of A defined as morphisms from **1** to A); *products* and *coproducts* of finite families of objects; *equalizers* and *coequalizers*; for any two objects there is an *exponential*; existence of a specific object **N** with morphisms $0\colon \mathbf{1} \to \mathbf{N}$ and $s\colon \mathbf{N} \to \mathbf{N}$ yielding the successor operation s on **N** and a simple recursion for sequences; axiom that **1** is a *generator* (if parallel morphisms f, g are not equal, then there is an element $x \in A$ such that $xf \neq xg$); axiom of choice; three additional elementary axioms of this sort (everything in the language of CT). This was augmented with one non-elementary axiom: $\mathcal{C}$ has products and coproducts for any *indexing infinite set*. A coproduct of copies of **1** played the role of a set (Lawvere, 1964; Mac Lane, 1986, pp.386–407; 1988, pp.341–345).

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[10] The oldest theorems of this type are: Cayley's theorem that every (abstract) group is isomorphic to a group of bijections of a set; Kuratowski's theorem that every partially ordered set is order-isomorphic to a family of subsets of a set, ordered by inclusion; theorem that every group with one free generator is isomorphic to $\mathbb{Z}$. Analogous examples are known in many theories.

The point was not to avoid membership relation completely, but (instead of taking as the starting point the primitive notions of *set*, *element* and membership $\in$) *function* may be taken as a primitive notion of the theory (with suitable axioms, using elementary logic, but avoiding any reference to sets) and then one derives membership and most concepts of set theory as a special case from there.

In the second half of the 1960's Lawvere opened a way to a new theory of *elementary toposes* (called also *elementary topoi*, with Greek plural $\tau\acute{o}\pi o\iota$ of the noun $\tau\acute{o}\pi o\varsigma$). Unexpected territories of mathematics were discovered (Lawvere, 1972; Mac Lane, 1988, pp.352–359; Krömer, 2007).

CT became a contender for a foundation of mathematics, although the hope that it undermine the overwhelming role of set theory turned out unfounded and most working mathematicians keep away from CT and toposes. CT yields new tools to study many formal mathematical theories and mutual relations between them, from a perspective different from that set theory.

6. Recapitulation of some points

Let us recall Atiyah's remark (quoted in the Introduction) that really important discoveries get later omitted altogether as they become absorbed by the general mathematical culture. This thought fits particularly well with the case of CT. Most of the ideas presented by Eilenberg and Mac Lane in 1945 have been absorbed as a natural language of advanced mathematical thinking. Once mathematicians learnt the definitions of a functor and a natural transformation, these concepts became a major tool of mathematical reasoning in many abstract theories of the second half of the 20^{th} century. However, it took several years to realise the scope of the change. Freyd commented as follows:

> MacLane's definition of "product" (1950) as the solution of a universal mapping problem was revolutionary. So revolutionary that it was not immediately absorbed even by most category minded people.
>
> [...] In a new subject it is often very difficult to decide what is trivial, what is obvious, what is hard, what is worth bragging about (Freyd, 1964, p.156).

Mac Lane, however, used the definition of a product and its dual only in the case of groups (general or abelian). He did not formulate it *mutatis mutandis* in the general case of a category, although he had several simple examples at hand. Freyd also told the story of the term *exact sequence*, a technical definition in homological algebra. In the late 1950's, when he was a graduate student at Brown University, he was brought up to think in terms of exactness of maps. This concept seemed to him as fundamental as the notion of continuity must seem to an analyst. And later he was astonished to hear that when Eilenberg and Steenrod wrote their fundamental book (1952) they defined this very notion, recognized the importance of the choice of a suitable name for it, yet could not invent any satisfactory word. Consequently, they wrote the word "blank" throughout most of the manuscript, ready to replace it before submitting the book for publication. After entertaining an unrecorded number of possibilities they settled on "exact" (Freyd, 1964, p.157).

One may argue that the 1945 definitions of a category and of a functor were within a major onward development of abstract algebra and other advanced topics. In fact, originally they were not regarded as a novelty. Eilenberg and Mac Lane were not even certain whether their paper will be accepted for publication (it was long and lacked theorems with substantial proofs). However, they were genuinely convinced of the significance of their conceptual clarification and took pains to write the paper clearly to attract the reader.

After this publication for almost ten years CT appeared dormant. The groundbreaking papers on abelian categories by Buchsbaum and Grothendieck marked a far-reaching change. And then—in the 1960's—CT unexpectedly started to grow rapidly, with astonishing results (Mac Lane, 1988, pp.338–339, 341–361).

Thus, from the present perspective, in spite of the previous arguments, one can say that the emergence of CT was undoubtedly a major transgression in mathematics. It was a crossing of a previously non-traversable barrier of deep-rooted habits to think of mathematics. A vivid argument is the fact that—even after publication of the main ideas—it was so difficult to overcome the previous inhibition and widespread tradition.[11]

The creators of CT and their followers could choose their definitions freely, nobody could forbid that. And yet the previous way of thinking was an obstacle for potential authors and for prospective readers. Great insight of Eilenberg and Mac Lane of what is significant in mathematics turned out a crucial factor.

[11] Many mathematicians, in USA and elsewhere, expressed disinclination about CT. Karol Borsuk, an outstanding topologist, the teacher of Eilenberg in Warsaw and coauthor of their joint paper published in 1936, was later critical of CT and the categorical methods in mathematics (Jackowski, 2015, p.30). Jerzy Dydak, a student of Borsuk, recalled after years: *My own PhD thesis written under Borsuk in 1975 makes extensive use of category theory and I was asked by him to cut that stuff out. Only after I assured him that I spent many months trying to avoid abstract concepts, he relinquished and the thesis was unchanged* (Dydak, 2012, p.92).

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